7 Advanced Counting Techniques

7.1 Recurrence Relations

1. a recurrence relation for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of previous term(s) of the sequence
2. a recurrence relation together with initial condition(s) determine a recursive definition of the sequence
3. recurrence relations model compound interest problems, population increase/decrease problems, determining the moves in the Tower of Hanoi puzzle...

7.2 Solving Recurrence Relations

linear homogeneous recurrence relation of degree k with constant coefficients

1. this is a recurrence relation that is linear (each term on the right hand side has a_i^1 for all $i \leq n-1$, versus containing quadratic terms like a_i^2), homogeneous (each term on the right hand side has some a_i for some $i \leq n-1$, versus containing constants or terms without a_i), it has degree k (a_n is defined in terms of a_j, where a_j is not more than k steps away from a_n, i.e. $n-k \leq j \leq n$), and the coefficients of each a_i are constants
2. the characteristic equation is obtained by replacing a_n by r^n in the recurrence relation (and canceling by the lowest power of r in the characteristic equation, one can solve it for r)
3. solutions are of the form $a_n = r^n$, or linear combinations of the roots to power n
4. to solve a linear homogeneous recurrence relation find the characteristic equations and solve for the characteristic roots of the recurrence, and then use a linear combination of the roots together with the initial condition to find solutions (if there are repetitions in the roots, then use the monomial n^j, where $1 \leq j \leq$ the multiplicity of the root -1): Theorem: If the characteristic equation has t roots r_i ($1 \leq i \leq t$), each of multiplicity m_i ($\sum_{i=1}^{t} m_i = k$), then a sequence $\{a_n\}$ is a solution of the given recurrence relation if and only iff

$$a_n = (\sum_{j=0}^{m_1-1} \alpha_j n^j) r_1^n + (\sum_{j=0}^{m_2-1} \beta_j n^j) r_2^n + \ldots$$

for $n = 0, 1, 2, \ldots$, where α_j, β_j are constants that you can find using the initial conditions.

nonlinear nonhomogeneous recurrence relation with constant coefficients

1. example: $a_n = 4a_{n-2} + 3n$, with $a_0 = 0, a_1 = 1$
2. a solution of a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots c_k a_{n-k} + F(n),$$

   (where $F(n)$ is a function of n) is the sum of a particular solution and a solution of the associated linear homogeneous recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots c_k a_{n-k}$
3. there is no general method in finding a particular solution, however there are techniques that work for certain types of functions $F(n)$:
   - if $F(n)$ is a polynomial $b_t n^t + b_{t-1} n^{t-1} + b_1 n + b_0$, then a particular root will be a polynomial of the same degree t (see Example 10 and Homework problem #25)
   - if $F(n)$ is an exponential s^n, then a particular root will be an exponential times a constant: $c \cdot s^n$ (see Example 11 and Homework problem #23)
   - if $F(n)$ is a polynomial times and exponential, then the particular solution is a combination of the two (see Theorem 6)

7.5 Inclusion-Exclusion

1. The principle of Inclusion-Exclusion: Let $A_1, A_2, \ldots, A_n$ be finite sets. Then

$$|A_1 \cup A_2 \cup \ldots A_n| = \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n+1} |A_1 \cap A_2 \cap \ldots \cap A_n|.$$

2. if $n = 2$: $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$ and if $n = 3$:

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_1 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

7.6 Applications of Inclusion-Exclusion

1. if we're counting the number $N(P_1', P_2', \ldots, P_n')$ of elements that <u>do not have properties P_i $(1 \leq i \leq n)$</u> we can use the following: let A_i be the subset counting the elements that have the property P_i $(1 \leq i \leq n)$, and so the number of elements without any properties P_i is $N - |A_1 \cup A_2 \cup \ldots A_n|$, where N is the total number of elements in the set (the value $|A_1 \cap A_2 \cap \ldots A_n|$ is denoted by $N(P_1, P_2, \ldots, P_n)$ and it represents the number of elements that have the properties P_i $(1 \leq i \leq n)$)

2. <u>The Sieve of Eratosthenes</u> is used to find all the primes not exceeding a specified positive integer n: list all the natural numbers between 2 and $n-1$ (inclusive), and then keep the first prime number but delete its multiples, then keep the second prime number but delete its multiples, ... up to the largest prime number that is less than or equal to n. To use the inclusion-exclusion principle: let P_1 be the statement that the first prime number divides n, P_2 be the statement that the second prime number divides n, ..., P_k be the statement that the kth prime number divides n (where the last prime number, k, is at most $\sqrt{n}$). Then

$$\begin{aligned} N(P_1'P_2'\ldots P_k') &= (n-1) - N(P_1) - N(P_2) - \ldots - N(P_k) \\ &+ N(P_1P_2) + N(P_1P_3) + N(P_2P_3) + \ldots N(P_{k-1}P_k) \\ &- N(P_1P_2P_3) - \ldots - N(P_{k-2}P_{k-1}P_k) \\ &+ N(P_1P_2P_3P_4)... \end{aligned}$$

3. <u>The number of onto functions</u> from a set with m elements to a set with n elements: if we let P_k denote the property that the value k is not in the range (i.e. there is no value x of the domain that gets mapped to k), then

$$\begin{aligned} N(P_1'P_2'\ldots P_k') &= N - N(P_1) - N(P_2) - \ldots - N(P_k) \\ &+ N(P_1P_2) + N(P_1P_3) + N(P_2P_3) + \ldots N(P_{k-1}P_k) \\ &- N(P_1P_2P_3) - \ldots - N(P_{k-2}P_{k-1}P_k) \\ &+ N(P_1P_2P_3P_4)... \\ &= n^m - C(n,1)(n-1)^m + C(n,2)(n-2)^m - \ldots + (-1)^{n-1}C(n,n-1)1^m \end{aligned}$$

4. a <u>derangement</u> is a permutation of n objects that leaves no objects in their original position (i.e. when permuting the elements, every element needs to change its position). The number of derangements of n elements is D_n (let P_i be the permutation that fixes element i $(1 \leq i \leq n)$, and count $D_n = N(P_1'P_2'\ldots P_n')$ using the inclusion-exclusion principle))

$$D_n = n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \binom{n}{3}(n-3)! + \ldots + (-1)^n\binom{n}{n}(n-n)!$$

$$D_n = n!\Big(1 - \tfrac{1}{1!} + \tfrac{1}{2!} - \tfrac{1}{3!} + \ldots + (-1)^n\tfrac{1}{n!}\Big)$$